Meeting 11 — Math 22

Section 10.8.1 Lagrange Multipliers (Continued...)

Recall: Suppose you want to optimize $f(x,y)$ subject to a constraint $g(x,y)=c$. If (x,y) optimizes f subject to constraint, there exists λ such

$$\nabla f(x,y) = \lambda \nabla g(x,y).$$

Example

Optimize: $f(x,y) = x^2y$ Constraint: $4x+y=108$
$g(x,y)=4x+y$

Solve the system of 3 equations for $(x,y),\lambda$:

$$\langle 2xy, x^2\rangle = \nabla f = \lambda\nabla g = \lambda\langle 4,1\rangle$$

$\Rightarrow$ ① $\frac{2xy}{4} = \lambda$ ② $x^2=\lambda$

③ $4x+y=108$

Eq. ① + ② $\frac{1}{2}xy = \lambda = x^2$

If $x\neq 0$, we can solve for y:

④ $y=2x$

Eq ④+③ $4x+2x=108 \Rightarrow x=18, y=36$

If $x=0$, then by eq. ③ $y=108$

There are two "critical points" $(18,36)$, $(0,108)$.
Evaluate f at both critical points:

$f(18,36) = 18^2\cdot 36 = 11{,}664$ → maximum

$f(0,108) = 0$ → minimum.

Lagrange Multipliers To optimize $f(x,y)$ subject to $g(x,y)=c$:

① Solve system of equations:
$$\nabla f(x,y) = \lambda\nabla g(x,y)$$
for (x,y), λ

② Evaluate f at (x,y) points from ①

③ The largest value is the max and the smallest value is the min.

Activity 10.8.2

- Complete w/ your group.
- Class discussion.

(a) The variables are the radius and the height of a cylinder. The radius and height are constrained by the volume.

(b) We want to minimize surface area

$$S(r,h) = 2\pi rh + 2\pi r^2$$

the constraint is $\pi r^2 h = V(r,h) = 355$.

(c) Solve
- $\nabla S(r,h) = \lambda\nabla V(r,h)$
- $\pi r^2 h = 355$

$\nabla S(r,h) = \langle 2\pi h + 4\pi r, 2\pi r\rangle$

$\lambda\nabla V(r,h) = \lambda\langle 2\pi rh, \pi r^2\rangle$

System of eq: ① $2\pi h + 4\pi r = \lambda\cdot 2\pi rh$
② $2\pi r = \lambda\pi r^2 \Rightarrow 2=\lambda r$
③ $\pi r^2 h = 355 \Rightarrow \pi h = \frac{355}{r^2}$

Notice:
- eq ③ implies $r,h\neq 0$ so we can divide by r or h.
- Eq ② implies $\lambda\neq 0$, so we can divide by λ.

Try to combine eq ①,②,③

$$\frac{710}{r^2} + 4\pi r = \frac{1420}{r^2}$$

$\Rightarrow 710 + 4\pi r^3 = 1420 \Rightarrow r^3 = \frac{710}{4\pi} = \frac{355}{2\pi}$

$\Rightarrow r = \left(\frac{355}{2\pi}\right)^{1/3}$

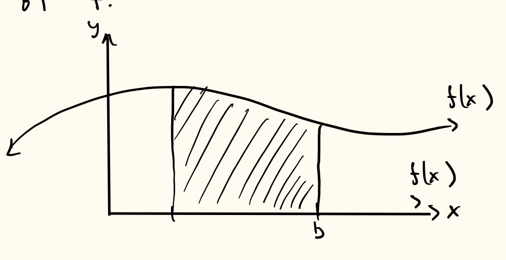
$\Rightarrow$ Using eq ③ $h = \frac{355}{\pi\left(\frac{355}{2\pi}\right)^{2/3}} = \frac{355}{\pi\cdot\frac{355^{2/3}}{2^{2/3}\pi^{2/3}}}$
$= \frac{\sqrt[3]{355}\cdot\sqrt[3]{4}}{\sqrt[3]{\pi}}$
$= \sqrt[3]{\frac{1420}{\pi}}$

At $\left(\sqrt[3]{\frac{355}{2\pi}}, \sqrt[3]{\frac{1420}{\pi}}\right)$ the surface area is minimized.

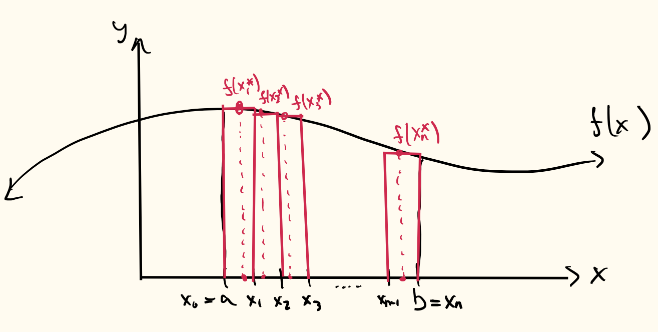
Section 11.1 Double Riemann Sums & Double Integrals over Rectangles

Review Riemann Sums from Single-Variable Calculus

Let $f(x)$ be a function that is continuous on $[a,b]$. The goal is to compute the (signed) area under the graph of f.



Step 1: Partition $[a,b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ w/ endpoints $x_0=a<x_1<x_2<\cdots<x_n=b$



Step 2. Within each subinterval $[x_{i-1},x_i]$ pick a "sample" point x_i^*. Form a rectangle w/ base $[x_{i-1},x_i]$ and height $f(x_i^*)$.

Step 3. The area of each rectangle is $f(x_i^*)\Delta x$. Form the Riemann sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

This approximates the area under the graph of f on $[a,b]$.

Step 4. Increase the number of subintervals n (take a limit). The definite integral is

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*)\Delta x$$

which is equal to the signed area under the curve on $[a,b]$.

Reading Debrief
- Discuss Activity 11.1.2 w/ your group.
- Questions?

(k) $\sum_{j=1}^{3}\sum_{i=1}^{4} f(x_{ij}^*, y_{ij}^*)\Delta A$

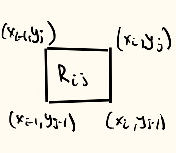
Section 11.1.2 Double Riemann Sums & Double Integrals

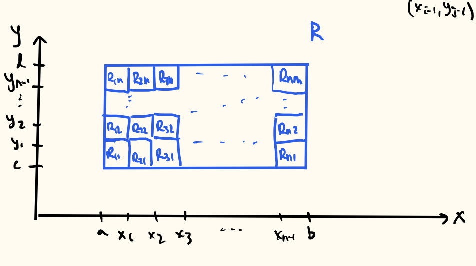
Let $f(x,y)$ be a function which is continuous on a rectangle $R=[a,b]\times[c,d]$.
A double Riemann sum is constructed as follows:

① Partition $[a,b]$ into m subintervals of equal length $\Delta x = \frac{b-a}{m}$ w/ endpoints $x_0=a<x_1<x_2<\cdots<x_m=b$.

② Partition $[c,d]$ into n subintervals of equal length $\Delta y = \frac{d-c}{n}$ w/ endpoints $y_0=c<y_1<\cdots<y_n=d$.

③ Together, ① and ② partition R into $m\cdot n$ rectangles $R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j]$ w/ areas $\Delta A = \Delta x\Delta y$

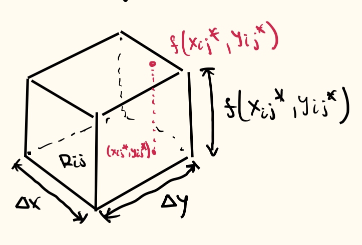




4. For each rectangle R_{ij}, choose a sample point (x_{ij}^*, y_{ij}^*) in R_{ij}. The double Riemann sum is

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Each quantity $f(x_{ij}^*, y_{ij}^*)\Delta A$ is the signed volume of a rectangular prism



The double Riemann sum approximates the signed volume of the solid bounded by the graph of f and the xy-plane on R.
By increasing the number of subrectangles we can obtain the precise volume.

Definition Let $f(x,y)$ be a continuous function on a rectangle $R=\{(x,y): a\le x\le b, c\le y\le d\}$. With the notation used above, the double integral of $f(x,y)$ over R is the limit

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty}\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_{ij}^*, y_{ij}^*)\Delta A$$

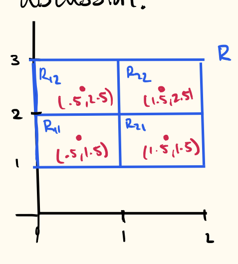
Section 11.1.3 Interpretations of Double Integrals

Activity 11.1.3

- Complete w/ your group.
- Class discussion.

(a)

(b)



$\Delta A = 1$

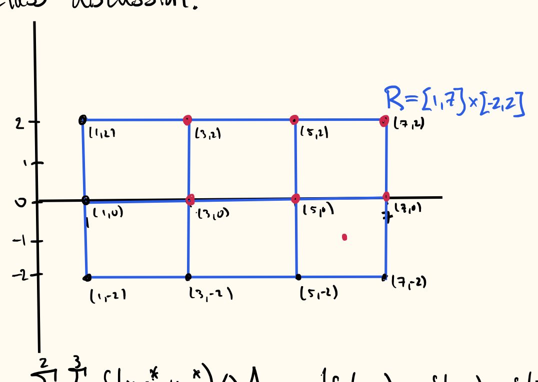
(c)
$$\sum_{j=1}^{2}\sum_{i=1}^{2} f(x_{ij}^*, y_{ij}^*)\Delta A = f(.5,1.5) + f(1.5,1.5) + f(.5,2.5) + f(1.5,2.5)$$
$= 3.5 + 4.5 + 5.5 + 6.5$
$= 20.$

(d) This is an approximation of the signed volume of the solid bounded by the function $f(x,y) = x+2y$ and the xy-plane on the rectangle $R=[0,2]\times[1,3]$.

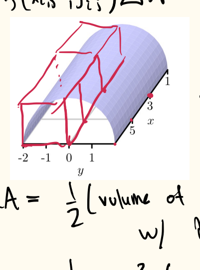
Activity 11.1.4

- Complete w/ your group.
- Class discussion.

(a)



(b)
$$\sum_{j=1}^{2}\sum_{i=1}^{3} f(x_{ij}^*, y_{ij}^*)\Delta A = \left(f(3,0) + f(5,0) + f(7,0) + f(3,2) + f(5,2) + f(7,2)\right)\cdot 4$$
$= (2+2+2+0+0+0)\cdot 4$
$= 24.$



(c) $\iint_R f(x,y)\,dA = \frac{1}{2}$(volume of cylinder w/ radius 2 w/ height 6)
$= \frac{1}{2}\pi\cdot 2^2\cdot 6 = 12\pi \approx 37.699.$